Section 5.6: Substitution and the Area Between Curves

Warm-Up Problems
(A) Integrate \( \int \sqrt{2x^2 + 4x(4x + 4)} \, dx \)

(B) Use an appropriate integral to find the shaded area under the curve \( f(x) = 4 - x^2 \).

Substitution with Definite Integrals

Recall: We use definite integrals to find the signed area between a curve and the \( x \)-axis. But we need to be careful when we evaluate definite integrals using substitution! Remember that the limits on the integral are in terms of \( x \)!

There are two methods to deal with this.
Method 1 for Definite Integrals with Substitution

**Method 1**: Transform the integral as though it is indefinite. Integrate, change back to $x$, and use the original $x$ limits.

Calculate: $\int_{0}^{2} \sqrt{2x^2 + 4x(4x + 4)} \, dx$

---

Method 2 for Definite Integrals with Substitution

**Method 2**: Transform the integral AND the limits of integration. Then evaluate the transformed integral with the transformed limits.

Calculate: $\int_{0}^{2} \sqrt{2x^2 + 4x(4x + 4)} \, dx$
Why does this work?

**Theorem 7: Substitution in Definite Integrals**

If \( g' \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( g(x) = u \), then

\[
\int_a^b f(g(x)) \cdot g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

**Proof:**

Another Example

**Example 2:** Find \( \int_{-\pi}^{\pi} \frac{-1}{4} \tan\left(\frac{x}{4}\right) dx \) using both methods.
(Hint: remember that \( \tan(x) = \frac{\sin(x)}{\cos(x)} \)).

**Method 1:**

**Method 2:**
Hmm.... The area is 0?

Notice: because of the symmetry of the function $f(x) = \tan(x)$, the area on the interval $[-\pi, 0]$ is the opposite of the area on the interval $[0, \pi]$, so over the interval $[-\pi, \pi]$, they sum to 0.

Watching out for tricks like this can save time when finding definite integrals!

---

Theorem 8: Definite Integrals on Symmetric Functions

**Theorem 8:** Let $f$ be continuous on the symmetric interval $[-a, a]$.

(a) If $f$ is even, then $\int_{-a}^{a} f(x)\,dx = 2 \int_{0}^{a} f(x)\,dx$

(b) If $f$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0$
Area Between Curves

**Question:** If this graph shows the functions $f(x) = 2 - \frac{1}{2}x^2$ and $g(x) = \frac{1}{2}x^2 - 2$, how could you find the total area of the shaded region between these two curves? Brainstorm with a partner.

Area Between Curves (continued)

If $f$ and $g$ are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from $a$ to $b$ is the integral of $(f(x) - g(x))$ from $a$ to $b$:

$$A = \int_a^b [f(x) - g(x)]\,dx$$
Example 3: Find the area between the curves. Recall the functions are \( f(x) = 2x + 4 \) and \( g(x) = x^2 - 4 \).

More practice

Example 4: Determine the definite integral(s) you would need to find the area of the regions bounded by \( f(x) = x^3 \) and \( g(x) = \frac{x}{4} \). Note: you do not need to calculate the area! Just set up the integral(s).
Finding Area Between Curves with Respect to $y$

**Question:** How can we find the area between two curves given as functions of $y$?

We can use the formula

$$A =$$

...to find the area of a region bounded by curves described as functions of $y$. Note that $f$ denotes the right-hand curve and $g$ denotes the left-hand curve, since we want $f(y) \geq g(y)$.

**Practice**

**Example 5:** Determine the definite integral(s) you would need to find the area of the regions bounded by $x = 2y^2 - 2y$ and $x = 8y^2 - 8y^3$. *Note: you do not need to calculate the area! Just set up the integral(s).*
Extra Practice Problems

Try these by yourself or with a partner.

**Extra Practice 1:**
\[ \int_{1}^{e} \frac{\ln x}{x} \, dx \]

**Extra Practice 2:**
\[ \int_{-4}^{4} (2x^7 - x^3 + \sin x) \, dx \]
Extra Practice Problems:

Extra Practice 3: Find the area of the shaded region.

\[ f(x) = (x^2 - 2x)^2(2x - 2) \]

Extra Practice Problems:

Extra Practice 4: Find the area of the shaded region.

\[ x = y^2 - 2 \]
\[ x = 6 - y^2 \]