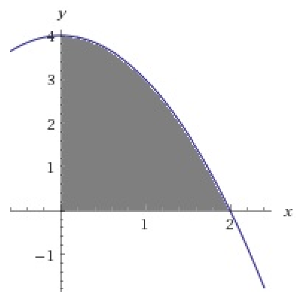
Section 5.6: Substitution and the Area Between Curves

Warm-Up Problems

(A) Integrate $\int \sqrt{2x^2 + 4x}(4x + 4)dx$

(B) Use an appropriate integral to find the shaded area under the curve $f(x) = 4 - x^2$.



Substitution with Definite Integrals

Recall: We use definite integrals to find the signed area between a curve and the x-axis. But we need to be careful when we evaluate definite integrals using substitution! Remember that the limits on the integral are in terms of x! There are two methods to deal with this.

Method 1 for Definite Integrals with Substitution

Method 1: Transform the integral as though it is indefinite. Integrate, change back to x, and use the original x limits.

Calculate: $\int_0^2 \sqrt{2x^2+4x}(4x+4)dx$

Method 2 for Definite Integrals with Substitution

Method 2: Transform the integral AND the limits of integration. Then evaluate the transformed integral with the transformed limits.

Calculate: $\int_0^2 \sqrt{2x^2+4x}(4x+4)dx$

Why does this work?

Theorem 7: Substitution in Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

Proof:

Another Example

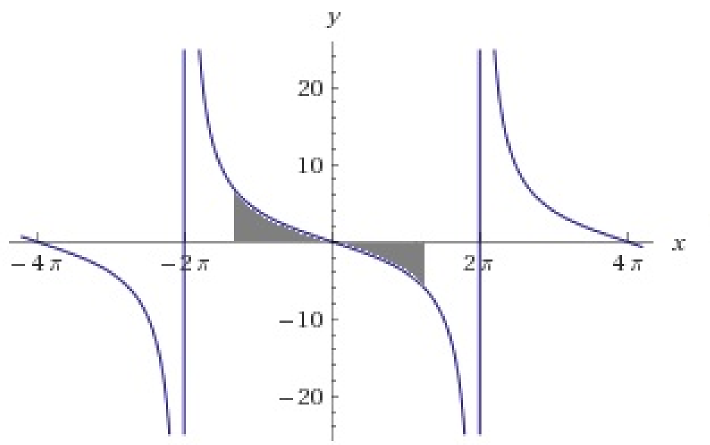
Example 2: Find $\int_{-\pi}^{\pi} \frac{-1}{4} \tan(\frac{x}{4}) dx$ using both methods. (Hint: remember that $\tan(x) = \frac{\sin(x)}{\cos(x)}$).

Method 1:

Method 2:

Hmm.... The area is 0?

Notice: because of the symmetry of the function $f(x) = \tan(x)$, the area on the interval $[-\pi, 0]$ is the opposite of the area on the interval $[0, \pi]$, so over the interval $[-\pi, \pi]$, they sum to 0.

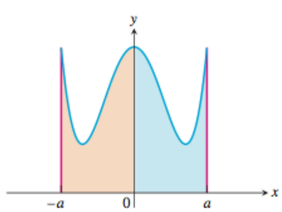


Watching out for tricks like this can save time when finding definite integrals!

Theorem 8: Definite Integrals on Symmetric Functions

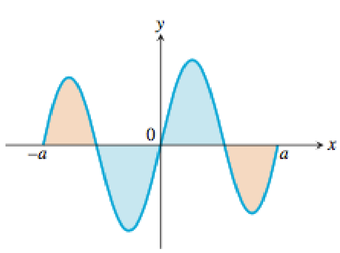
Theorem 8: Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$



(a)

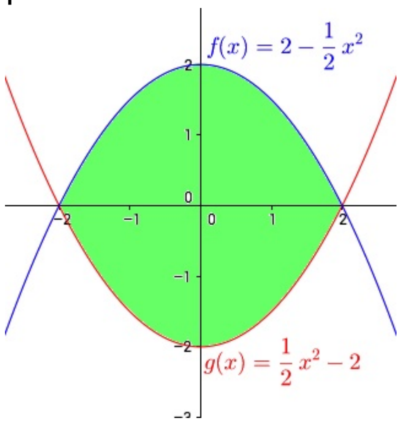
(b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$



(b)

Area Between Curves

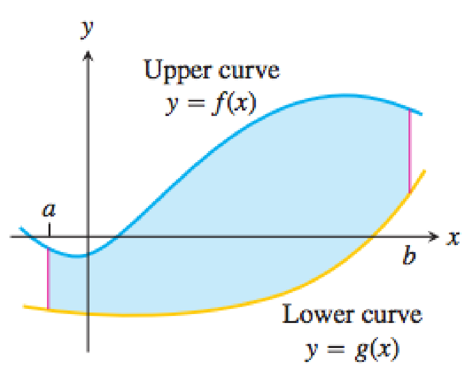
Question: If this graph shows the functions $f(x) = 2 - \frac{1}{2}x^2$ and $g(x) = \frac{1}{2}x^2 - 2$, how could you find the total area of the shaded region between these two curves? Brainstorm with a partner.



Area Between Curves (continued)

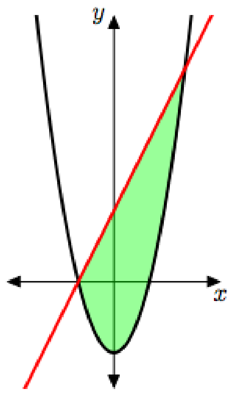
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f(x) - g(x))$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx$$



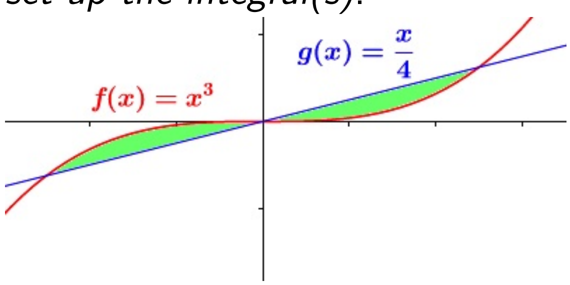
Practice

Example 3: Find the area between the curves. Recall the functions are $f(x) = 2x + 4$ and $g(x) = x^2 - 4$.



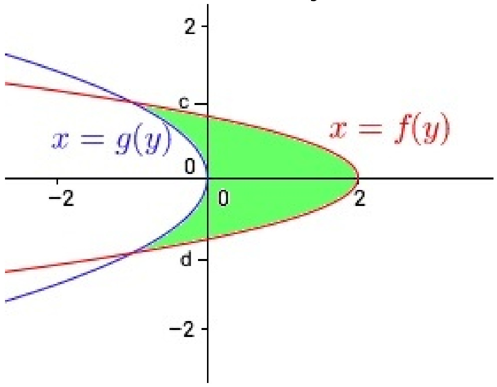
More practice

Example 4: Determine the definite integral(s) you would need to find the area of the regions bounded by $f(x) = x^3$ and $g(x) = \frac{x}{4}$. *Note: you do not need to calculate the area! Just set up the integral(s).*



Finding Area Between Curves with Respect to y

Question: How can we find the area between two curves given as functions of y?



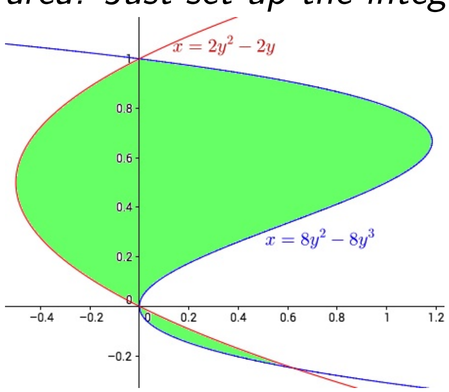
We can use the formula

$$A =$$

to find the area of a region bounded by curves described as functions of y. Note that f denotes the **right-hand curve** and g denotes the **left-hand curve**, since we want $f(y) \geq g(y)$.

Practice

Example 5: Determine the definite integral(s) you would need to find the area of the regions bounded by $x = 2y^2 - 2y$ and $x = 8y^2 - 8y^3$. *Note: you do not need to calculate the area! Just set up the integral(s).*



Extra Practice Problems

Try these by yourself or with a partner.

Extra Practice 1:

$\int_1^e \frac{\ln x}{x} dx$

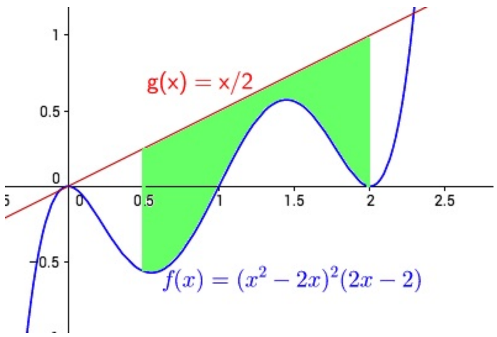
Extra Practice Problems

Extra Practice 2:

$\int_{-4}^4 (2x^7 - x^3 + \sin x)dx$

Extra Practice Problems:

Extra Practice 3: Find the area of the shaded region.



Extra Practice Problems:

Extra Practice 4: Find the area of the shaded region.