Recursion Theoretic Results for the Game of Cops and Robbers on Graphs

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Throughout, $G = (V, E)$ is assumed to be a connected reflexive graph with no double-edges.

- In the game of *Cops and Robbers*, there are two players: a single robber, $R$, and a cop, $C$.
- The game is played in rounds, beginning with the cop $C$ occupying a certain vertex, followed by the robber choosing a vertex to occupy.
- In each round, the cop moves first, followed by the robber. A move consists of a player moving to any vertex that is adjacent to their current vertex.
- The cop wins if after some finite number of moves, he occupies the same vertex as the robber. The robber wins if he can evade capture indefinitely.
A winning strategy for the cop is a set of rules that results in a win for the cop, regardless of the strategy the robber uses. If a winning strategy for a cop exists for a given graph $G$, we say $G$ is *cop-win*.

**Example:**
In the following cop-win graph $G$, the cop has a winning strategy of moving to vertex $e$, and then moving to whatever vertex $R$ chooses to occupy in the next round.
Winning Strategies

A graph that is not cop-win is defined to be \textit{robber-win}. A winning strategy for the robber is a set of rules that allows the robber to evade capture indefinitely, regardless of the strategy the cop uses. If a winning strategy for the robber exists for a given graph $G$, it is \textit{robber-win}.

\textbf{Example:}

In the following cop-win graph $G$, the robber has a winning strategy by starting at the vertex opposite $C$, and always moving to a vertex distance 2 from the cop.
The following classes of graphs are cop-win for every \( n \):

- \( P_n \), a path of length \( n \).

- \( W_n \), a wheel on \( n \) vertices (i.e., an \( n \)-cycle along with one universal vertex).

- All finite trees.
Cops and Robbers on Infinite Trees

Theorem ([2])

The following are equivalent:

1. $T$ is a cop-win tree.
2. $T$ is a tree with no infinite paths.

Note: this is provable over RCA$_0$, but we can form alternate characterizations of this theorem that are not.
We say a graph $G$ is **locally finite** if every $v \in V$ is connected to only finitely many other nodes.

ACA$_0$ \iff every locally finite infinite tree is robber win.

There is a locally finite infinite tree for which every robber strategy computes $0'$.

A locally finite graph with $V = \{v_i : i \in \mathbb{N}\}$ is **highly locally finite** if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n$, if $E(v_n, v_m)$ holds, then $m \leq f(n)$.

WKL$_0$ \iff every highly locally finite infinite tree is robber win.

Every computable highly locally finite infinite tree has a low robber-win strategy.
Note that every locally finite infinite graph contains an infinite chordless path. Furthermore, $0'$ can compute such a path, since for every $n$ the set of vertices distance $n$ from the cop is computable from $0'$. 

Thus every locally finite infinite graph is robber-win, and this theorem is equivalent to ACA$_0$. 

If we restrict this theorem to highly locally finite infinite graphs, it is equivalent to WKL$_0$. 
Characterizing Cop-Win Graphs

In order to characterize Cop-Win Graphs of arbitrary size, we can use the following relation $\leq$ on the vertices of $G$. We define $\leq$ recursively on ordinals as follows:

- For all $v \in G$, $v \leq_0 v$.
- For $\alpha \in \mathbb{ON}$, let $u \leq_\alpha v$ if and only if for every $x \in N[u]$ there exists $y \in N[v]$ such that $x \leq_\beta y$ for some $\beta < \alpha$.
- Since $\alpha \leq \beta$ implies $\leq_\alpha \subseteq \leq_\beta$ as relations, and because these relations are bounded above in cardinality, there exists an ordinal $\rho$ such that $\leq_\rho = \leq_\rho + 1$. We choose the least such $\rho$ and define $\leq = \leq_\rho$. 
Theorem (Nowakowski, Winkler [3])

A graph $G$ is cop-win if and only if the relation $\preceq$ on $G$ is trivial.

- If $\preceq$ is not trivial, then we have $u \not\preceq v$ for some $u, v \in G$. Suppose the cop begins at $v$, and robber at $u$.

- The cop may choose to move to any neighbor $v_1$ of $v$. But by the definition of $\preceq = \leq_{\rho}$, there exists $u_1 \in N[u]$ such that for all $x \in N[v]$, we have $u_1 \not\preceq x$. Otherwise, we would have $u \leq_{\rho+1} v$, a contradiction.

- So the robber can move to $u_1$ and evade the cop. We now have $R = u_1 \not\preceq v_1 = C$, and so by induction the robber can always evade the cop for another round.
Characterizing Cop-Win Graphs

Theorem (Nowakowski, Winkler [3])

A graph $G$ is cop-win if and only if the relation $\preceq$ on $G$ is trivial.

• $\iff$ Suppose $\preceq$ is trivial. Say $R = u_0 \preceq v_0 = C$, with $\preceq = \leq \rho$. Then there must be some $v_1 \in N[v_0]$ and $\rho_1 < \rho$ such that $u_0 \leq \rho_1 v_1$.

• Suppose after $i$ rounds we have the the robber occupying $u_i$ and the cop occupying $v_i$ such that $u_i \leq \rho_i v_i$. Once again the cop can move to some $v_{i+1}$ such that $u_i \leq \rho_{i+1} v_{i+1}$ for some $\rho_{i+1} < \rho_i$.

• This yields a decreasing sequence of $\rho_i$'s. Since the ordinals are well-ordered, this sequence cannot be infinite and so $\rho_j = 0$ for some finite $j$. Then $u_j = v_j$ and the cop has won. $\square$
A graph $G$ is cop-win if and only if the relation $\preceq$ on $G$ is trivial.

A memoryless strategy is a function $f : V \times V \rightarrow V$, i.e. a strategy which takes into account only the current position of the cop and robber. The $\preceq$ relation implies the existence of a memoryless cop-win strategy for cop-win graphs.
Question: If we require that cops and robbers play with computable strategies on computable graphs, does the characterization of cop-win (and robber-win) trees and graphs still hold?
Theorem

There exists a computable graph that is classically robber-win, such that no computable robber strategy is a winning strategy.

Proof: We have seen the existence of a locally finite infinite tree such that each winning robber strategy computes $0'$. 
Classically cop-win graphs with no computable cop-win strategy

**Theorem**

*There exists a computable cop-win graph such that no computable memoryless cop-strategy is a winning strategy.*

**Proof:** We construct such a graph $G$ in stages to diagonalize against every possible computable strategy $\varphi_e$. Begin with $G_0$ as follows:

![Diagram of the construction](image)
Classically cop-win graphs with no computable cop-win strategy

If at a stage \( s > e \) we see \( \varphi_e(C_e, R_e) \downarrow = x_e \), we add in vertices \( a_0 \) and \( b_0 \) as follows:
Classically cop-win graphs with no computable cop-win strategy

If at a later stage $t > s > e$ we see $\varphi_e(x_e, a_0) \downarrow = b_0$ or $R_e$, we add in vertices $a_1$ and $b_1$ as follows:

We continue building the graph in this fashion, and let $G = \cup G_e$. 
Why is this graph cop-win?

- If there are only finitely many $a_i$ and $b_i$ vertices for a given $C_e, x_e, R_e$ path, then the cop can win by moving to the highest index $b_i$, since that vertex is adjacent to all other vertices.
- If there is an infinite path of $a_i$ vertices and $b_i$ vertices and the robber starts at some $a_i$, $b_i$, $R_e$ or $x_e$, the cop can win by moving from $C_e$ to $b_{i+1}$. 

Recursion Theoretic Results for the Game of Cops and Robbers on Graphs 18 / 27 Shelley Stahl
Why will no computable cop strategy be a winning one?

- If there are only finitely many $a_i$ and $b_i$ vertices for a given $C_e, x_e, R_e$ path, then $\varphi_e$ gave up on chasing down the robber.
- If there is an infinite path of $a_i$ vertices and $b_i$ vertices, we know the cop will make the wrong choice infinitely many times.
Can we find cop-win strategies that are arbitrarily complex?

- In the last example, no cop strategy was computable.
- Can we construct a cop-win graph such that every cop-win strategy computes $0'$?
Existence of winning cop strategies of relatively low complexity

Theorem

Suppose $G$ is a computable infinite cop-win graph, and $A$ is a non-computable set. If $\{r_i : i \in \omega\}$ is a countable set of robber strategies, then there is a history cop-strategy $c$ such that $c \not\geq_T A$, and $c$ is a winning strategy against each $r_i$.

- An allowable play sequence for $G$ is a finite sequence of vertices $\sigma = \langle c_0, r_0, c_1, r_1, \cdots, r_n \rangle$, beginning with an initial cop position and satisfying $c_{i+1} \in N[c_i]$ and $r_{i+1} \in N[r_i]$ for all $i < n$. Note that if $G$ is computable, the set of allowable play sequences is computable.
- The proof of this relies on building a cop-win strategy $F = \bigcup F_e$ using forcing conditions $F_e$, finite functions from the set of allowable sequences to $V$, to satisfy:
  - $R_e$: $\Phi_e^F \neq A$
  - $P_e$: $F$ yields a cop strategy that beats $r_e$
Existence of winning cop strategies of relatively low complexity

- Assume $F_{s-1}$ is a forcing condition. To satisfy $R_e$, define $F_s$ as follows:
  - If $\exists x \Phi^F_e(x) \uparrow$ for all cop strategies $F$ extending $F_{s-1}$, set $F_s = F_{s-1}$.
  - If there exists some $x$ and some forcing condition $F'$ extending $F_{s-1}$ such that $\Phi^F_{e'}(x) \downarrow \neq A(x)$, set $F_s = F'$

- Note that we must be in one of these two cases; otherwise, $A$ is in fact computable.
Existence of winning cop strategies of relatively low complexity

- Assume $F_{s-1}$ is a forcing condition. To satisfy $P_e$, first define a memoryless cop strategy $c \preceq (v_i, v_j) = v_k$ for $v_j \leq \alpha v_i$, where $k$ is the least index for $v \in N[v_i]$ s.t. $v_j \leq \beta v$ for some $\beta < \alpha$. Now start a game in which the robber follows $r_e$, and the cop follows $F_{s-1}$ as long as possible.
  - If $F_{s-1}$ is defined enough to result in a win for the cop, define $F_s = F_{s-1}$.
  - Otherwise, extend $F_{s-1}$ to $F_s$, defined on an allowable play sequence in which the cop follows $c \preceq$ and the robber follows $r_e$.

- Note that $F_s$ will still be finite, as $c \preceq$ will give the cop a strategy to win in finitely many moves.
Existence of winning cop strategies of relatively low complexity

Now define $F = \bigcup F_e$. Then $F$ yields a cop strategy $c$ that wins against each $r_e$, and such that $c \not\geq_T A$. □
Further Questions to study

- Can we find a global cop-win strategy (history or memoryless) that does not compute a given non-computable set $A$?
- Do there exist infinite robber-win trees that require strategies above $0'$, or in general above $0^{(\alpha)}$?
- How complex are the sets $\leq_\alpha$ in general?
Thank you!
Slides available at wp.rachel-stahl.grad.uconn.edu
References


