Recursion Theoretic Results for the Game of Cops and Robbers on Graphs

Shelley Stahl

University of Connecticut

N.E.R.D.S. November 2016

Games on graphs background

Throughout, $G = (V, E)$ is assumed to be a connected reflexive graph with no double-edges.

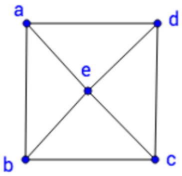
- In the game of *Cops and Robbers*, there are two players: a single robber, R, and a cop, C.
- The game is played in rounds, beginning with the cop C occupying a certain vertex, followed by the robber choosing a vertex to occupy.
- In each round, the cop moves first, followed by the robber. A move consists of a player moving to any vertex that is adjacent to their current vertex.
- The cop wins if after some finite number of moves, he occupies the same vertex as the robber. The robber wins if he can evade capture indefinitely.

Winning Strategies

A winning strategy for the cop is a set of rules that results in a win for the cop, regardless of the strategy the robber uses. If a winning strategy for a cop exists for a given graph G, we say G is *cop-win*.

Example:

In the following cop-win graph G, the cop has a winning strategy of moving to vertex e, and then moving to whatever vertex R chooses to occupy in the next round.

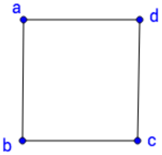


Winning Strategies

A graph that is not cop-win is defined to be *robber-win*. A winning strategy for the robber is a set of rules that allows the robber to evade capture indefinitely, regardless of the strategy the cop uses. If a winning strategy for the robber exists for a given graph G, it is *robber-win*.

Example:

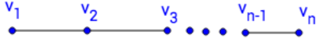
In the following cop-win graph G, the robber has a winning strategy by starting at the vertex opposite C, and always moving to a vertex distance 2 from the cop.



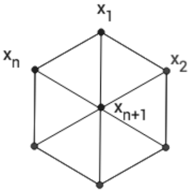
Cop-Win Finite Graphs

The following classes of graphs are cop-win for every n:

- P_n, a path of length n.



- W_n, a wheel on n vertices (i.e., an n-cycle along with one universal vertex).



- All finite trees.

Cops and Robbers on Infinite Trees

Theorem ([2])

The following are equivalent:

- *(1) T is a cop-win tree.*
- *(2) T is a tree with no infinite paths.*

Note: this is provable over RCA_0, but we can form alternate characterizations of this theorem that are not.

(Highly) Locally Finite Trees

- We say a graph G is **locally finite** if every $v \in V$ is connected to only finitely many other nodes.
- $\mathsf{ACA}_0 \Leftrightarrow$ every locally finite infinite tree is robber win.
- There is a locally finite infinite tree for which every robber strategy computes $\mathbf{0}'$
- A locally finite graph with $V = \{v_i : i \in \mathbb{N}\}$ is **highly locally finite** if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every n, if $E(v_n, v_m)$ holds, then $m \leq f(n)$.
- $\mathsf{WKL}_0 \Leftrightarrow$ every highly locally finite infinite tree is robber win.
- Every computable highly locally finite infinite tree has a low robber-win strategy.

Characterization of Locally Finite Graphs

- Note that every locally finite infinite graph contains an infinite chordless path. Furthermore, $\mathbf{0}'$ can compute such a path, since for every n the set of vertices distance n from the cop is computable from $\mathbf{0}'$.
- Thus every locally finite infinite graph is robber-win, and this theorem is equivalent to ACA_0.
- If we restrict this theorem to highly locally finite infinite graphs, it is equivalent to WKL_0.

Characterizing Cop-Win Graphs

In order to characterize Cop-Win Graphs of arbitrary size, we can use the following relation $\preceq$ on the vertices of G. We define $\preceq$ recursively on ordinals as follows:

- For all $v \in G$, $v \leq_0 v$.
- For $\alpha \in \mathbb{ON}$, let $u \leq_\alpha v$ if and only if for every $x \in N[u]$ there exists $y \in N[v]$ such that $x \leq_\beta y$ for some $\beta < \alpha$.
- Since $\alpha \leq \beta$ implies $\leq_\alpha \subseteq \leq_\beta$ as relations, and because these relations are bounded above in cardinality, there exists an ordinal ρ such that $\leq_\rho = \leq_{\rho+1}$. We choose the least such ρ and define $\preceq = \leq_\rho$.

Characterizing Cop-Win Graphs

Theorem (Nowakowski, Winkler [3])

A graph G is cop-win if and only if the relation $\preceq$ on G is trivial.

- $\Rightarrow$ If $\preceq$ is not trivial, then we have $u \not\preceq v$ for some $u, v \in G$. Suppose the cop begins at v, and robber at u.
- The cop may choose to move to any neighbor v_1 of v. But by the definition of $\preceq = \leq_\rho$, there exists $u_1 \in N[u]$ such that for all $x \in N[v]$, we have $u_1 \not\preceq x$. Otherwise, we would have $u \leq_{\rho+1} v$, a contradiction.
- So the robber can move to u_1 and evade the cop. We now have $R = u_1 \not\preceq v_1 = C$, and so by induction the robber can always evade the cop for another round.

Characterizing Cop-Win Graphs

Theorem (Nowakowski, Winkler [3])

A graph G is cop-win if and only if the relation $\preceq$ on G is trivial.

- $\Leftarrow$ Suppose $\preceq$ is trivial. Say $R = u_0 \preceq v_0 = C$, with $\preceq = \leq_\rho$. Then there must be some $v_1 \in N[v_0]$ and $\rho_1 < \rho$ such that $u_0 \leq_{\rho_1} v_1$.
- Suppose after i rounds we have the the robber occupying u_i and the cop occupying v_i such that $u_i \leq_{\rho_i} v_i$. Once again the cop can move to some v_{i+1} such that $u_i \leq_{\rho_{i+1}} v_{i+1}$ for some $\rho_{i+1} < \rho_i$.
- This yields a decreasing sequence of ρ_i's. Since the ordinals are well-ordered, this sequence cannot be infinite and so $\rho_j = 0$ for some finite j. Then $u_j = v_j$ and the cop has won. $\square$

Characterizing Cop-Win Graphs

Theorem (Nowakowski, Winkler [3])

A graph G is cop-win if and only if the relation $\preceq$ on G is trivial.

- A memoryless strategy is a function $f : V \times V \to V$, i.e. a strategy which takes into account only the current position of the cop and robber. The $\preceq$ relation implies the existence of a memoryless cop-win strategy for cop-win graphs.

Computability Results for Infinite Graphs

Question: If we require that cops and robbers play with computable strategies on computable graphs, does the characterization of cop-win (and robber-win) trees and graphs still hold?

Computability Results for Infinite trees

Theorem

There exists a computable graph that is classically robber-win, such that no computable robber strategy is a winning strategy.

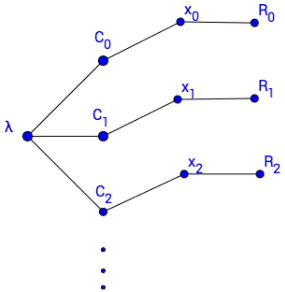
Proof: We have seen the existence of a locally finite infinite tree such that each winning robber strategy computes $\mathbf{0}'$.

Classically cop-win graphs with no computable cop-win strategy

Theorem

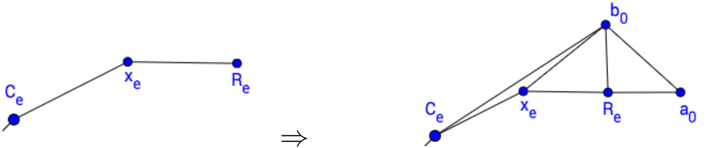
There exists a computable cop-win graph such that no computable memoryless cop-strategy is a winning strategy.

Proof: We construct such a graph G in stages to diagonalize against every possible computable strategy φ_e. Begin with G_0 as follows:



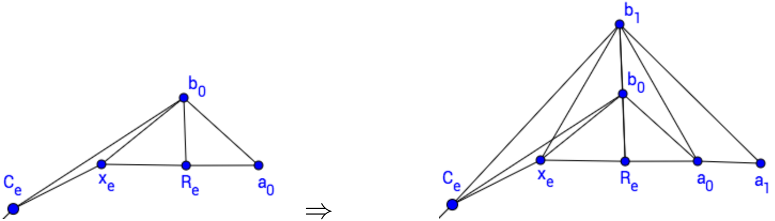
Classically cop-win graphs with no computable cop-win strategy

If at a stage $s > e$ we see $\varphi_e(C_e, R_e) \downarrow = x_e$, we add in vertices a_0 and b_0 as follows:



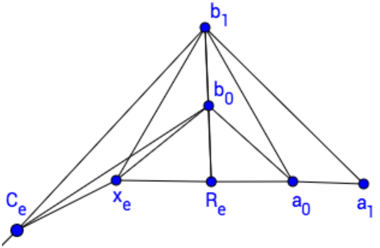
Classically cop-win graphs with no computable cop-win strategy

If at a later stage $t > s > e$ we see $\varphi_e(x_e, a_0) \downarrow = b_0$ or R_e, we add in vertices a_1 and b_1 as follows:



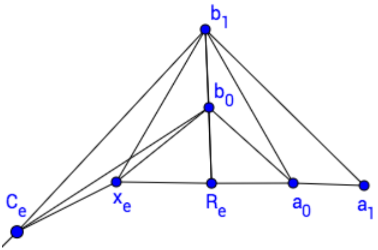
We continue building the graph in this fashion, and let $G = \cup G_e$.

Why is this graph cop-win?



- If there are only finitely many a_i and b_i vertices for a given C_e, x_e, R_e path, then the cop can win by moving to the highest index b_i, since that vertex is adjacent to all other vertices.
- If there is an infinite path of a_i vertices and b_i vertices and the robber starts at some a_i, b_i, R_e or x_e, the cop can win by moving from C_e to b_{i+1}.

Why will no computable cop strategy be a winning one?



- If there are only finitely many a_i and b_i vertices for a given C_e, x_e, R_e path, then φ_e gave up on chasing down the robber.
- If there is an infinite path of a_i vertices and b_i vertices, we know the cop will make the wrong choice infinitely many times.

Can we find cop-win strategies that are arbitrarily complex?

- In the last example, no cop strategy was computable.
- Can we construct a cop-win graph such that every cop-win strategy computes $\mathbf{0}'$?

Existence of winning cop strategies of relatively low complexity

Theorem

Suppose G is a computable infinite cop-win graph, and A is a non-computable set. If $\{r_i : i \in \omega\}$ is a countable set of robber strategies, then there is a history cop-strategy c such that $c \not\geq_T A$, and c is a winning strategy against each r_i.

- An *allowable play sequence* for G is a finite sequence of vertices $\sigma = \langle c_0, r_0, c_1, r_1, \cdots, r_n \rangle$, beginning with an initial cop position and satisfying $c_{i+1} \in N[c_i]$ and $r_{i+1} \in N[r_i]$ for all $i < n$. Note that if G is computable, the set of allowable play sequences is computable.
- The proof of this relies on building a cop-win strategy $F = \cup F_e$ using forcing conditions F_e, finite functions from the set of allowable sequences to V, to satisfy:
  - R_e: $\Phi_e^F \neq A$
  - P_e: F yields a cop strategy that beats r_e

Existence of winning cop strategies of relatively low complexity

- Assume F_{s-1} is a forcing condition. To satisfy R_e, define F_s as follows:
  - If $\exists x \Phi_e^F(x) \uparrow$ for all cop strategies F extending F_{s-1}, set $F_s = F_{s-1}$.
  - If there exists some x and some forcing condition F' extending F_{s-1} such that $\Phi_e^{F'}(x) \downarrow \neq A(x)$, set $F_s = F'$
- Note that we must be in one of these two cases; otherwise, A is in fact computable.

Existence of winning cop strategies of relatively low complexity

- Assume F_{s-1} is a forcing condition. To satisfy P_e, first define a memoryless cop strategy $c_{\preceq}(v_i, v_j) = v_k$ for $v_j \leq_\alpha v_i$, where k is the least index for $v \in N[v_i]$ s.t. $v_j \leq_\beta v$ for some $\beta < \alpha$. Now start a game in which the robber follows r_e, and the cop follows F_{s-1} as long as possible.
  - If F_{s-1} is defined enough to result in a win for the cop, define $F_s = F_{s-1}$.
  - Otherwise, extend F_{s-1} to F_s, defined on an allowable play sequence in which the cop follows $c_{\preceq}$ and the robber follows r_e.
- Note that F_s will still be finite, as $c_{\preceq}$ will give the cop a strategy to win in finitely many moves.

Existence of winning cop strategies of relatively low complexity

- Now define $F = \cup F_e$. Then F yields a cop strategy c that wins against each r_e, and such that $c \not\geq_T A$. $\square$

Further Questions to study

- Can we find a global cop-win strategy (history or memoryless) that does not compute a given non-computable set A?
- Do there exist infinite robber-win trees that require strategies above $\mathbf{0}'$, or in general above $\mathbf{0}^{(\alpha)}$?
- How complex are the sets $\leq_\alpha$ in general?

Thank you!
Slides available at wp.rachel-stahl.grad.uconn.edu

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